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16

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13. ABSTRACT

This paper describes a procedure for determining if constrained transportation problems (i.e., transportation problems with additional linear constraints) can be transformed into equivalent pure transportation problems by a linear transformation involving the node constraints and the extra constraints. Our results extend procedures for problems in which the extra constraints consist of bounding certain partial sums of variables.

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1.0 INTRODUCTION

The classical transportation problem is well known for its widespread applicability and for the facility with which it can be solved. Many procedures have been developed for reformulating disparate linear programming problems as transportation problems to take advantage of the computational efficiency inherent in the specialized transportation algorithm [2,3,8,10]. We generalize the concepts of an earlier paper [6] to show how constrained transportation problems (i.e., transportation problems with additional linear constraints) can be transformed into pure transportation problems. Our procedure determines if an arbitrary extra linear constraint can be transformed into an equivalent bounded partial sum of variables involving a single node constraint. If this is possible the procedure gives the linear transformation that yields the equivalent constraint. This extends the work of Wagner [9], Manne [4,p. 382], and Charnes [1] who have shown how transportation problems with these bounded partial sums can be reformulated into pure transportation problems

It is conjectured that our transformation requires computational effort on the same order as that required to find an initial basic solution by Vogel's Approximation Method. Since computational results indicate that specialized transportation codes can solve transportation problems at least 150 times faster than general purpose linear programming codes [5], our results make it possible to solve constrained transportation problems of the specified class with substantially greater efficiency than by a general purpose algorithm.

2.0 PROBLEM STATEMENT, MATHEMATICAL DEVELOPMENT, AND AN EXAMPLE

A transportation problem with one additional extra constraint (a singularly constrained transportation problem) can be stated mathematically as follows:

$$\text{Minimize} \quad \sum_{i\varepsilon M} \sum_{j\varepsilon N} c_{ij}\, x_{ij}$$

subject to

$$\sum_{j\varepsilon N} x_{ij} = a_i, \quad i\varepsilon M$$

$$\sum_{i\varepsilon M} x_{ij} = b_j, \quad j\varepsilon N$$

$$\sum_{i\varepsilon M} \sum_{j\varepsilon N} p_{ij}x_{ij} = d$$

$$x_{ij} \geq 0 \qquad , i\varepsilon M,\ j\varepsilon N$$

where $M = \{1,2,\dots,m\}$, $N = \{1,2,\dots n\}$, $\sum_{i\varepsilon M} a_i = \sum_{j\varepsilon N} b_j$. In detached coefficient form the singularly constrained transportation problem appears as indicated in Table 1. The coefficients (p_{ij}) of the extra constraint appear below the familiar echelon-diagonal structure of ones of the transportation problem.

coefficient of a linear combination	$x_{11}x_{12}\dots x_{1n}$ $c_{11}c_{12}\dots c_{1n}$	$x_{21}x_{22}\dots x_{2n}$ $c_{21}c_{22}\dots c_{2n}$	$\dots$	$x_{m1}x_{m2}\dots x_{mn}$ $c_{m1}c_{m2}\dots c_{mn}$		variables minimize
R_1	1 1 ... 1				=	a_1
R_2		1 1 ... 1			=	a_2
⋮			⋮		⋮	
R_m				1 1 ... 1	=	a_m
K_1	1	1	...	1	=	b_1
K_2	1	1	...	1	=	b_2
⋮	⋱	⋱		⋱	⋮	
K_n	1	1	...	1	=	b_n
1	$p_{11}p_{12}\dots p_{1n}$	$p_{21}p_{22}\dots p_{2n}$	$\dots$	$p_{m1}p_{m2}\dots p_{mn}$	=	d
case a		1 1			=	f
case b		1		1	=	f
case c	1 1				=	f

Table 1. Detached Coefficient Form of Singularly Constrained Transportation Problems

Our goal is to specify a computationally simple procedure for identifying a linear combination of the ordinary transportation constraints, if one exists, which can be subtracted from the extra constraint to produce an inequality that is equivalent to establishing a bound for a partial sum of variables associated with a single origin or destination. From well known properties of the transportation matrix, if there exists a linear combination of the transportation constraints that has the desired form, then a linear combination can be found in which any particular transportation constraint receives a zero weight. Thus, we may arbitrarily delete a node constraint when seeking such a linear combination. Having done this certain variables can be viewed as having only one entry in the coefficient matrix for purposes of finding the desired linear combination. To exploit these facts in seeking a transformation of the singularly constrained transportation problem into an ordinary transportation problem, we arbitrarily omit the first origin constraints. Our principal observations are then contained in the following three cases.

Case a. Assume that the extra constraint is equivalent to a partial sum of variables in a single origin constraint q other than origin 1 (i.e., the equivalent extra constraint is of the form $\sum_{j\varepsilon S} x_{qj}=f$ for $S\subset N$ as illustrated in Table 1). Having set R_1 equal to 0, the unique values for all the destination constraint multipliers can be determined using the equations $R_1+K_j=p_{1j}$. This is possible by the assumption that the equivalent constraint involves variables in origin $q\neq 1$ (i.e., the coefficient on the variables x_{1j}, $j=1,\dots, n$ in the equivalent constraint must equal 0). By the same reasoning a unique origin multiplier can be found for every other origin except origin q. In origin q not all of the equations $R_q + K_j = p_{qj}$ can simultaneously be satisfied by a single value for R_q. There will be two values, one that satisfies the equation for a subset $S\subset N$ and another that satisfies that equation for the destinations in the subset N-S. By setting R_q equal to the value $R_q + K_j = p_{qj}$ for $j\varepsilon N-S$ a linear combination of the standard node constraints has been found which when subtracted from the original extra constraint yields a restriction on the partial sum of variables in origin constraint q associated with the destinations in S. The equivalent constraint $\sum_{j\varepsilon S} (p_{qj}-R_q-K_j)\, x_{qj}=$ $d-\sum_{i\varepsilon M} R_i a_i - \sum_{j\varepsilon N} K_j b_j$ can be reduced to an equivalent partial sum by dividing through by $(p_{qj}-R_q - K_j)$ since this expression has the same value for all $j \epsilon S$.

Case b. Assume that the extra constraint is equivalent to a partial sum of variables in a single destination q (i.e. the constraint is of the form $\sum_{i \epsilon T} x_{iq} = f$ for $T \subset M$ as illustrated in Table 1). The appropriate linear combination for this case can be found in the same manner as in case a. After the origin constraint multiplier R_1 is set equal to zero and the destination constraint multipliers are determined using the equations $R_1 + K_j = p_{1j}$, the remaining R_i values can be found. In every origin i that does not include variables in the equivalent partial sum (i.e. for $i \epsilon M-T$), there exists a unique value for R_i such that the constraints $R_i + K_j = p_{ij}$ will be satisfied simultaneously by a unique value for all $j \epsilon N$. For those origins $i \epsilon T$ that have a variable in common with the equivalent extra constraint, the equation $R_i + K_j = p_{ij}$ will be satisfied by a unique value for all $j \neq q$. Setting R_i, $i \epsilon T$ equal to this unique value will provide the scalar multipliers for a linear combination of the node constraints which when subtracted from the original extra constraint yield an equivalent partial sum of variables in the single destination constraint q associated with the origins in T. The equivalent constraint $\sum_{i \epsilon T} (p_{iq} - R_i - K_q) x_{iq} = d - \sum_{i \epsilon M} R_i a_i - \sum_{j \epsilon N} K_j b_j$ can be reduced to an equivalent partial sum by dividing through by $(p_{iq}-R_i - K_q)$ since this expression has the same value for all $i \epsilon T$.

Case c. Assume the extra constraint is equivalent to a partial sum of variables in origin constraint 1 (i.e. the constraint is $\sum_{j \epsilon S} x_{1j} = f$ for $S \subset N$ as illustrated in Table 1). Starting again with R_1 equal to zero, the K_j values for $j \epsilon N$ are immediately determined. Since the equivalent partial sum is assumed to be in origin 1, multipliers must be found to satisfy $R_i + K_j = p_{ij}$

for all i and j such that $i \neq 1$ in order for these variables not to appear in the equivalent constraint. However given the current values for K_j unique values for R_i, $i \neq 1$ cannot be found, for if they could be then $R_i + K_j = p_{ij}$ would hold for all i and all j contradicting the assumption. Thus there must be two possible values for each R_i, $i \neq 1$. Set these R_i equal to the value that satisfies $R_i + K_j = p_{ij}$ for $j \epsilon N-S$. Thus, $R_i + K_j \neq p_{ij}$ for $i \neq 1$ and for $j \epsilon S$. Note, however, new values for K_j can be found that satisfy $R_i + K_j = p_{ij}$ for $i \neq 1$ and $j \epsilon S$. Thus by changing the appropriate K_j values after having determined values for all R_i, one can find the linear combination of the node constraints which when subtracted from the extra constraint leaves the equivalent restriction $\sum_{j \epsilon S} (p_{1j}-K_j)\, x_{1j} = d - \sum_{i \epsilon M} R_i A_i - \sum_{j \epsilon N} K_j b_j$. Since $(p_{1j}-K_j)$ is the same for all $j \epsilon S$ then dividing through by this quantity will yield the equivalent partial sum.

Based on the reasoning just presented we can describe a general procedure to effect the transformation as required in the three cases. It is convenient to use a transportation tableau with the coefficients of the extra constraint placed in the cell corresponding to the appropriate variable. This format is illustrated in Table 2. Values for the R_i and K_j multipliers are shown around the rim of the tableau and can be determined in a manner similar to that used to find values for dual evaluators for a basic solution to a transportation problem.

	K_1	K_2	$\cdots$	K_n	
R_1	p_{11}	p_{12}	$\cdots$	p_{1n}	a_1
R_2	p_{21}	p_{22}	$\cdots$	p_{2n}	a_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
R_m	p_{m1}	p_{m2}	$\cdots$	p_{mn}	a_m
	b_1	b_2	$\cdots$	b_n	

Table 2. Tableau Representation of an Extra Constraint

Our procedure for finding the appropriate linear combination of the constraints is:

Step 1. Set R_1 equal to zero.

Step 2. For all $j \varepsilon N$ set K_j equal to p_{1j}.

Step 3. Try to determine a unique value for R_2 using the equations

$R_2 + K_j = P_{2j}$ for all $j \varepsilon N$.

a) If a unique value for R_2 can be found set R_2 equal to this value and proceed to step 4.

b) If the equations are not satisfied by a unique value of R_2 but all equations except q can be satisfied by a single value of R_2, set R_2 so that the $R_2 + K_q \neq p_{2q}$ and mark cell (2,q) with a "star". However, if this is not the case, and R_2 must assume two distinct values to satisfy the equations for all j, set R_2 equal to either of two values arbitrarily and "star" the cells for which $R_2 + K_j \neq p_{2j}$. Proceed to step 4.

c) If more than two values of R_2 are required to satisfy all the equations, stop. The constraint is not equivalent to a partial sum of variables in a single node constraint.

Step 4. Continue determining values for the remaining R_i as in step 2 except when R_i must assume two distinct values in order for the equations $R_i + K_j = p_{ij}$ to be satisfied for all j. In this case set R_i so that the starred cells in this row lie in the same columns as those starred in earlier rows. If this cannot be done, stop. The constraint is not equivalent to a partial sum of variables in a single node constraint. Also for any column r with starred cells check to see that $p_{ir} - R_i = p_{kr} - R_k$ for $i \neq k$ where i and k are rows containing the starred cells. If this is not the case for all $i \neq k$, stop. Again the constraint is not equivalent to a partial sum of variables in a single node constraint.

Step 5. After all R_i have been determined, four cases are possible.

i) the starred cells occur only in a single row.

ii) the starred cells occur in all cells in a subset of the columns except for the cells in row 1.

iii) the starred cells occur only in a single column.

iv) the starred cells occur in some cells in a subset of the columns but not in row 1 and not in some other row.

In case i the starred cells indicate the variables included in the equivalent partial sum of variables. For these starred cells the coefficient on these variables in an equivalent constraint is $p_{ij} - R_i - K_j$. After forming the linear combination from the extra constraint only those terms will remain. Since $p_{ij} - R_i - K_j$ will be the same for all starred cells, the equivalent partial sum can be obtained by dividing through by the coefficient on the variables.

In case ii the K_j values for the columns containing starred cells can be changed so that the equation $R_i + K_j = p_{ij}$ holds for the starred cells. The effect is to "erase" the stars from these cells and place them in the cells in row 1 in these columns. This is now the same as case i.

In case iii the starred cells indicate the variables in a single destination constraint that comprise an equivalent constraint. Since $p_{ij} - R_i - K_j$ will be the same for all starred cells the equivalent partial sum can be obtained by dividing through by the coefficient of the variables.

In case iv the constraint is not equivalent to a partial sum of variables in a single origin or destination constraint.

For cases i, ii, and iii we have found the coefficients of the variables in an equivalent constraint. The new right-hand side value can be found by subtracting $\sum_{i \in M} R_i a_i + \sum_{j \in N} K_j b_j$ from the original right-hand side.

We can summarize our results to this point in the following theorem.

Theorem: If an extra constraint is equivalent by a linear transformation to a partial sum of variables in a single node constraint then the stated procedure finds the equivalent partial sum.

Proof: The hypothesis allows us to assert that the equivalence can be determined by a linear combination of the node constraints of the transportation problem. The remainder of the proof is contained in cases a,b, and c above.

A four origin, five destination constrained transportation problem is shown in Table 3 with the coefficients of the extra constraint indicated in the cells corresponding to the appropriate variables The associated supply and demand values are shown along the rim of the table. Assume that the extra constraint is an "equality" constraint with a right-hand side value of 84.

						Supply
	0	1	0	4	-5	10
	6	7	4	8	-1	15
	5	6	5	9	0	10
	-1	0	-1	3	-6	10
Demand	8	7	9	6	15	

Table 3

By applying the procedure described above, the following R_i and K_j multipliers are obtained.

In step 1 R_1 is set equal to zero.

In step 2 the K_j multipliers are set equal to the following values

$K_1 = 0,\ K_2 = 1,\ K_3 = 0,\ K_4 = 4,\ K_5 = -5$

In step 3 a unique value for R_2 cannot be found to satisfy $R_2 + K_j = p_{2j}$ for all j. The value 6 satisfies that equation for p_{21} and p_{22} and the value 4 satisfies that equation for p_{23}, p_{24} and p_{25}. Arbitrarily set R_2 equal to 6 and star the cells p_{23}, p_{24}, and p_{25}.

As required by step 4 R_3 is set equal to 5 because this value satisfies the equation $R_3 + K_j = P_{3j}$ for all j. Similarly a value of $R_4 = -1$ satisfies $R_4 + K_j = P_{4j}$ for all j. The results of the procedure to this point are shown in Table 4.

	$K_1 = 0$	$K_2 = 1$	$K_3 = 0$	$K_4 = 4$	$K_5 = -5$	Supply
$R_1 = 0$	0	1	0	4	-5	10
$R_2 = 6$	6	7	4*	8*	-1*	15
$R_3 = 5$	5	6	5	9	0	10
$R_4 = -1$	-1	0	-1	3	-6	10
Demand	8	7	9	6	15	

Table 4

To complete the transformation next compute the values of $(p_{ij} - R_i - K_j)$ for the starred cells, check that they are equal, and obtain the coefficients for the variables in an equivalent constraint. In particular the left-hand side of the equivalent constraint is $-2x_{23} + -2x_{24} + -2x_{25}$. The new right-hand side value is found by the formula $(+84) - \sum_{i=1}^{4} R_i a_i - \sum_{j=1}^{5} K_j b_j = 84 - (130) - (-44) = -2$. Dividing through by -2, we obtain the equivalent partial sum $x_{23} + x_{24} + x_{25} = 1$. Thus, by using the procedure described above we have found a partial sum of variables in a single row equivalent to the original extra constraint. The original transportation problem can be enlarged by one source and one destination in the manner suggested by Wagner [9], and an optimal solution to the original problem can be found using the transportation algorithm.

3.0 EXTENSIONS

In the development of the procedure for transforming extra constraints and in the example the original extra constraint was assumed to be an "equality" type. It should be clear that the same transformation can be made for both "less than or equal" and "greater than or equal" constraints. If there are several extra constraints then the procedure can be applied to each one separately to obtain an equivalent partial sum for each extra constraint. Wagner [9] has shown that if these partial sums involve disjoint sets of variables and if the sets are nested in the same node constraints then the problem can be transformed into an enlarged transportation problem.

4.0 APPLICATIONS

Many models have the structure of a transportation problem with additional restrictions. The extra constraints may represent secondary objectives or restrictions that are not reflected in the standard node constraints. To illustrate a typical example of this class of problems, consider the transportation model where warehouses supply markets and the objective is the standard one of finding a shipping pattern that will minimize the total shipping cost. Suppose additionally that the products shipped are of a fragile nature and if they are sent via particular routes each item must be specially packaged to prevent losses in shipping. Table 6 shows the packaging time in minutes per unit required to prepare a unit for shipment on the various routes. Suppose that we wish to limit the average packaging time per unit to at most 2-1/2 minutes per unit (i.e., for the 60 units that must be shipped, the total packaging time must not exceed 150 minutes). Using our procedure a linear combination of the node constraints of the transportation problems can be found which when subtracted from the original extra constraint yields an equivalent partial sum of variables $x_{22}+ x_{24} \leq 5$. The node constraint multipliers for the linear combination are indicated in table 6. Thus the

	$K_1 = 0$	$K_2 = 1$	$K_3 = 0$	$K_4 = 2$	$K_5 = 1$	Supply
$R_1 = 0$	0	1	0	2	1	10
$R_2 = 1$	1	3*	1	4*	2	20
$R_3 = 3$	3	4	3	5	4	20
$R_4 = 2$	2	3	2	4	3	10
Demand	15	10	5	5	25	60 Total

Table 6

2-1/2 minute average packaging time restriction can only be satisfied if the total number of units shipped along routes (2,2) and (2,4) is less than or equal to 5. The transportation problem can be transformed to include this restriction directly,and thus the original constrained transportation problem can be solved as a transportation problem witn one additional origin and one additional destination.

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